Closing *Tues*: 3.10

Closing *Thurs*: 4.1(1) and 4.1(2)

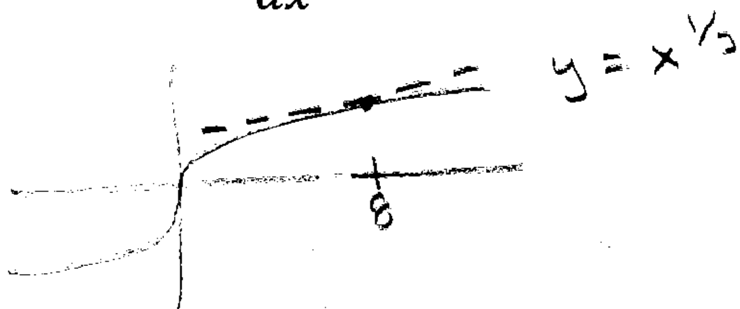
3.10 Linear Approx. *(continued)*

Recall:

Given a point (x_0, y_0) and a curve. The tangent line at the point can be thought of as a linear approximation:

$$y = m(x - x_0) + y_0$$

where $m = \frac{dy}{dx}$ at the point.



"ACTUAL" VALUE IS

2.0408275S1

Entry Task:

Using tangent line approximation to estimate the value of $\sqrt[3]{8.5}$.

Note the function is $f(x) = \sqrt[3]{x}$.

Use the "nice" nearby value of x.

$f(x) = \sqrt[3]{x} = x^{1/3}$

$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$

AT $x = 8$, $f(8) = \sqrt[3]{8} = 2$

$f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$

$y = \frac{1}{12}(x - 8) + 2$

THUS, $\sqrt[3]{x} \approx \frac{1}{12}(x - 8) + 2$ for $x \approx 8$

$\Rightarrow \sqrt[3]{8.5} \approx \underbrace{\frac{1}{12}(8.5 - 8)}_{1/24} + 2$

$\frac{1}{24} + 2 = 2.041\overline{6}$

ASIDE

$y = \frac{1}{12}(x - 8) + 2 \Rightarrow y - 2 = \frac{1}{12}(x - 8)$

LET $\left.\begin{array}{l} dy = y - 2 \\ dx = x - 8 \end{array}\right\}$ $dy = \frac{1}{12} dx$

Example (from HW):
A cone with height h and base radius r has total surface area:

$$S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

You start with $h = 8$ and $r = 6$, and you want to change the dimensions in such a way that the total *surface area remains constant*.

Suppose the height increases by 26/100.

In this problem, use tangent line approximation to estimate the new value of r so that the new cone has the *same total surface area*.

WE WANT TO APPROXIMATE r IF h CHANGES!

$$r = \square\,(h - \square) + \square$$

$\frac{dr}{dh} = ???$ 8   6

$$0 = 2\pi r \frac{dr}{dh} + \pi \frac{dr}{dh}\sqrt{r^2+h^2} + \pi r \frac{1}{2\sqrt{r^2+h^2}}\left(2r\frac{dr}{dh} + 2h\right)$$

$$h = 8,\ r = 6 \Rightarrow \sqrt{6^2+8^2} = \sqrt{100} = 10$$

$$\Rightarrow 0 = 12\pi \frac{dr}{dh} + 10\pi \frac{dr}{dh} + \underbrace{6\pi \frac{1}{20}}_{3\pi/10}\left(12\frac{dr}{dh} + 16\right)$$

$$0 = 22\pi\, dr/dh + \frac{36\pi}{10}\frac{dr}{dh} + \frac{48\pi}{10}$$

$$0 = \underbrace{\left(22\pi + \frac{18}{5}\pi\right)}\frac{dr}{dh} + \frac{24\pi}{5}$$

$$\Rightarrow \frac{128\pi}{5}\frac{dr}{dh} = -\frac{24\pi}{5}$$

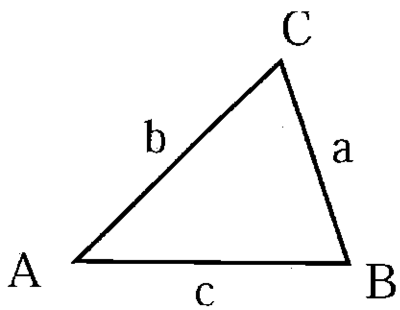
$$\Rightarrow \frac{dr}{dh} = -\frac{24\pi}{5}\cdot\frac{5}{128\pi} = -\frac{3}{16}$$

$$\boxed{r = -\frac{3}{16}(h - 8) + 6}$$

$8 + \frac{26}{100}$

Some Homework Hints:

Problem 10: Suppose that *a* and *b* are pieces of metal which are hinged at *C*.



By the ``law of sines,'' you *always* have:

$$\frac{b}{a} = \frac{\sin(B)}{\sin(A)}$$

At first: angle *A* is $\pi/4$ radians= 45^o and
angle *B* is $\pi/3$ radians = 60^o.

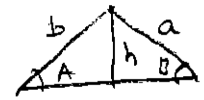
You then widen *A* to 46^o, without changing the sides *a* and *b*.

The question asks you to use the linear approximation to estimate the new angle B.

$$\underbrace{46^\circ = 46\,\frac{\pi}{180}}\ \text{RADIANS}$$

ASIDE

PROOF



$\sin(A) = \frac{h}{b}$ AND $\sin(B) = \frac{h}{a}$

$\Rightarrow h = b\sin(A)$ AND $h = a\sin(B)$

$\Rightarrow b\sin(A) = a\sin(B) \Rightarrow \frac{b}{a} = \frac{\sin(B)}{\sin(A)}$ //

CONSTANTS

$b\sin(A) = a\sin(B)$

WE WANT THE TANGENT LINE

$$B = \underset{dB/dA}{\underset{??}{\square}}\,(A - \underset{\substack{A_0\\ \pi/4}}{\underset{??}{\square}}) + \underset{\substack{B_0\\ \pi/3}}{\underset{?!}{\square}}$$

$b\cos(A) = a\cos(B)\frac{dB}{dA}$

$\Rightarrow \frac{dB}{dA} = \frac{b\cos(A)}{a\cos(B)}$

- At $B = \pi/3$ and $A = \pi/4$, WE HAVE

$$\frac{b}{a} = \frac{\sin(\pi/3)}{\sin(\pi/4)} = \frac{\sqrt{3}/2}{\sqrt{2}/2} = \sqrt{\frac{3}{2}}$$

AND $$\frac{dB}{dA} = \sqrt{\frac{3}{2}}\,\frac{\cos(\pi/4)}{\cos(\pi/3)} = \sqrt{\frac{3}{2}}\,\frac{\sqrt{2}/2}{1/2} = \sqrt{3}$$

$$\Rightarrow B = \sqrt{3}\,(A - \pi/4) + \pi/3$$

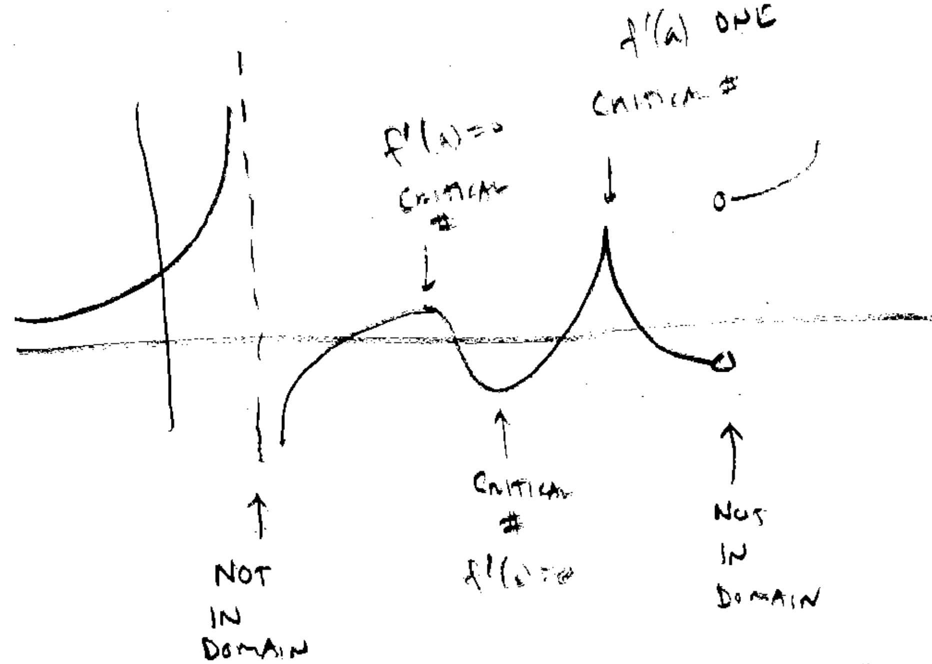
CONVERT BACK TO DEGREES

4.1: Critical Points and Absolute Max/Min

Given $y = f(x)$.
The first questions we always ask:

1. What is the domain?
   (What inputs are allowed?)

2. What are the "critical numbers"?
   A **critical number** is a number *x = a* that is <u>in the domain</u> and either
   (a) $f'(a) = 0$, or
   (b) $f'(a)$ does not exist.



$\frac{1}{x-2} \rightarrow$ DOMAIN $x \neq 2$

$\sqrt{x+3} \rightarrow$ DOMAIN $x \geq -3$

$\ln(x) \rightarrow$ DOMAIN $x > 0$.

Example (from homework):

$$y = x^3 + 3x^2 - 72x$$

a) What is the domain? $\boxed{\text{ALL REAL \#'S}}$

b) What are the critical numbers?

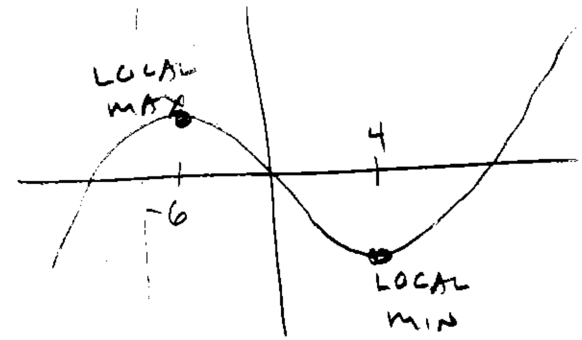
$\boxed{\text{ASIDE}}$

$$y' = 3x^2 + 6x - 72 \overset{?}{=} 0$$

$$3(x^2 + 2x - 24) \overset{?}{=} 0$$

$$3(x+6)(x-4) \overset{?}{=} 0$$

$$\boxed{\begin{aligned} x &= -6 \\ x &= 4 \end{aligned}}$$ CRITICAL NUMBERS



Example:

$$f(x) = 4x + \frac{1}{x} \quad = 4x + x^{-1}$$

a) What is the domain? $\boxed{x \neq 0}$

b) What are the critical numbers?

$$f'(x) = 4 - x^{-2} = 4 - \frac{1}{x^2}$$

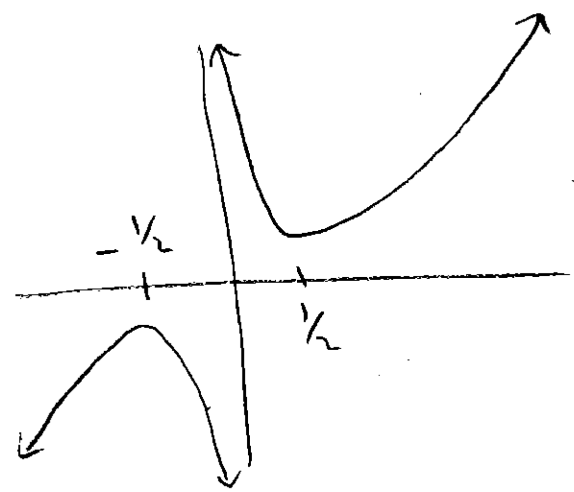
NOTE: $f'(x)$ DNE AT $x=0$, but x=0 IS NOT IN DOMAIN. (NOT A "CRITICAL NUMBER")

$$4 - \frac{1}{x^2} \stackrel{?}{=} 0$$

$$4x^2 - 1 = 0$$

$$x^2 = 1/4$$

$$\boxed{x = \pm 1/2}$$



Example:

$$g(x) = 3x - x^{1/3} = 3x - \sqrt[3]{x}$$ ← ODD ROOT

a) What is the domain? $\boxed{\text{ALL REAL NUMBERS}}$

b) What are the critical numbers?

$$g'(x) = 3 - \frac{1}{3}x^{-2/3} = 3 - \frac{1}{3x^{2/3}}$$

$g'(x)$ DNE AT $x=0$ AND $x=0$ IS IN THE DOMAIN.

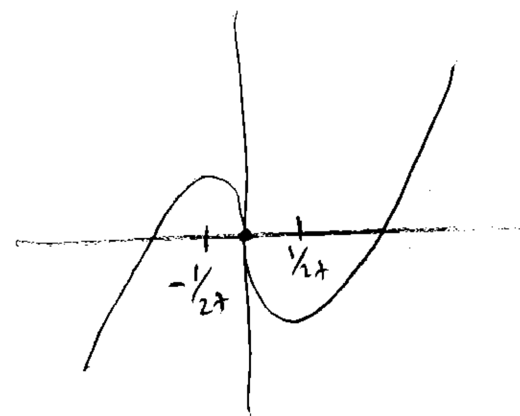
$\boxed{x=0 \text{ IS A CRITICAL NUMBER}}$ VERTICAL TANGENT

$$3 - \frac{1}{3x^{2/3}} \stackrel{?}{=} 0$$

$$9x^{2/3} - 1 = 0$$

$$x^{2/3} = 1/9$$

$$\boxed{x = \pm\left(1/9\right)^{3/2} = \pm\frac{1}{27}}$$ CRITICAL #'s



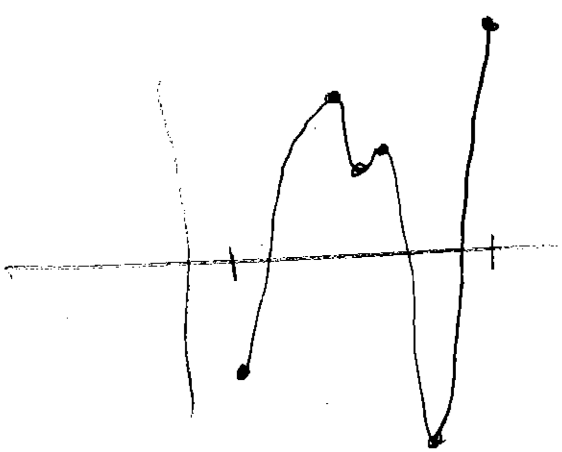
Absolute Max/Min

The **absolute max** (or **global** max) is the highest y-value on the interval.
The **absolute min** (or **global** min) is the lowest y-value on the interval.

Procedure to find absolute max/min:

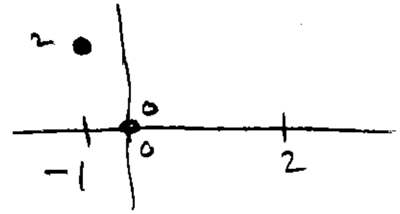
1. Find critical numbers.
2. Plug endpoints and critical numbers into the function.

<u>**Big, key, awesome observation**</u>:
(*Extreme Value Theorem*)
The absolute max/min always occur at critical numbers or endpoints!



Example (like HW):

Find the abs. max and min of $f(x) = x^3 + 3x^2$ on $[-1,2]$.



$$f'(x) = 3x^2 + 6x \overset{?}{=} 0$$

$$3x(x+2) = 0$$

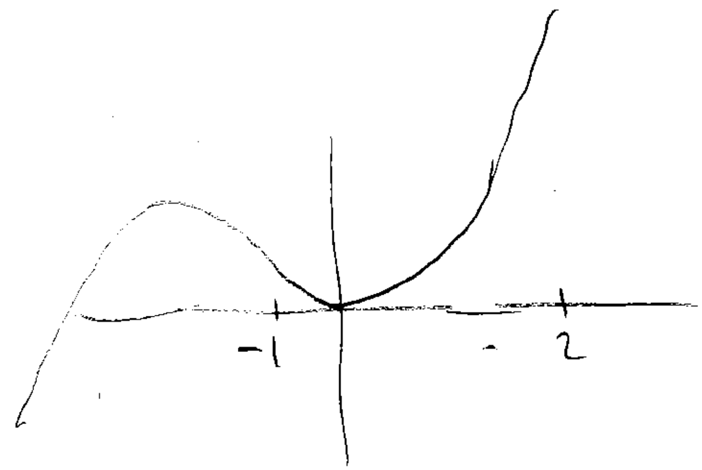
$$x = 0 \quad x = -2$$

ONLY CRITICAL # IN THIS DOMAIN

$$f(-1) = (-1)^3 + 3(-1)^2 = -1 + 3 = 2$$

$$f(0) = 0$$

$$f(2) = (2)^3 + 3(2)^2 = 8 + 12 = 20$$



ABS MAX = 20

ABS. MIN = 0

Small Note:

The **value** of a function, $y = f(x)$, is the output y-value. A question asking for the absolute max of a function is asking for the **y-value**. (The x-value is the location where the max *occurs*)

Example:

DOMAIN $x > 0$

Find the abs. max and min of $f(x) = x\ln(x)$ on $[1, e]$.

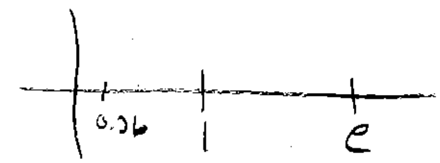
$$f'(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x)$$

$$1 + \ln(x) \stackrel{?}{=} 0$$

$$\ln(x) = -1$$

$$x = e^{-1} \approx 0.367879$$

NOT IN GIVEN DOMAIN!

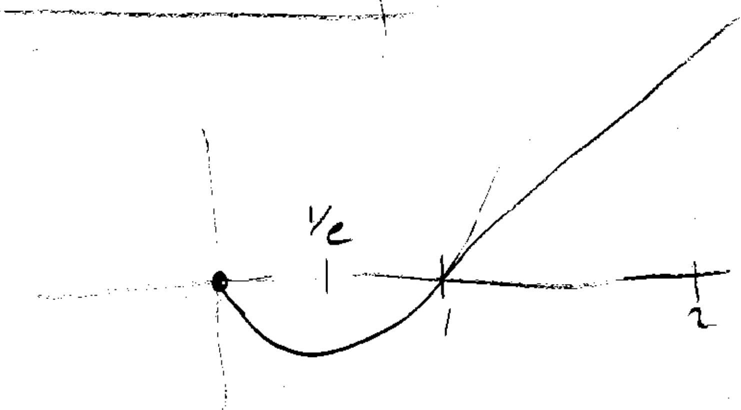


$f(1) = 1 \cdot \ln(1) = 0$

$f(e) = e\ln(e) = e$

ABS. MAX $= e$ occurs at $x=1$

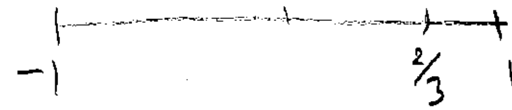
ABS. MIN $= 0$ occurs at $x=e$



Example:

DOMAIN $x \le 1$

Find the abs. max and min of
$f(x) = x\sqrt{1-x}$ on $[-1,1]$.



$$f'(x) = \sqrt{1-x} + x\,\frac{-1}{2\sqrt{1-x}} \overset{?}{=} 0$$

$$1 - x - \frac{1}{2}x = 0$$

$$1 - \frac{3}{2}x = 0$$

$$1 = \frac{3}{2}x$$

$$\boxed{x = 2/3}$$

Critical pts

$f'(x)$ DNE AT $\boxed{x = 1}$ ← IN DOMAIN.

$$f(-1) = (-1)\sqrt{1-(-1)} = -\sqrt{2} \approx -1.414$$

$$f(2/3) = \frac{2}{3}\sqrt{1 - 2/3} = \frac{2}{3}\sqrt{1/3}$$

$$= \frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{3}$$

$$\approx 0.3849$$

$$f(1) = (1)\sqrt{1-1} = 0$$

ABS. MAX $= \dfrac{2}{3\sqrt{3}}$

ABS MIN $= -\sqrt{2}$